ON REPRESENTATION ZETA FUNCTIONS OF GROUPS AND A CONJECTURE OF LARSEN AND LUBOTZKY

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Abstract. We study zeta functions enumerating finite-dimensional irreducible complex linear representations of compact p-adic analytic and of arithmetic groups. Using methods from p-adic integration, we show that the zeta functions associated to certain p-adic analytic pro-p groups satisfy functional equations. We prove a conjecture of Larsen and Lubotzky regarding the abscissa of convergence of arithmetic groups of type A_2 defined over number fields, assuming a conjecture of Serre on lattices in semisimple groups of rank greater than 1.

1. Introduction and statement of results

Let G be a group and, for $n \in \mathbb{N}$, denote by $r_n(G)$ the number of equivalence classes of n-dimensional irreducible complex representations of G; if G is a topological or an algebraic group, it is tacitly understood that representations are continuous or rational, respectively. We assume henceforth that G is (representation) *rigid*, i.e. that $r_n(G)$ is finite for all $n \in \mathbb{N}$. In the subject of representation growth one investigates the arithmetic properties of the sequence $r_n(G)$ and its asymptotic behaviour as n tends to infinity. Recent key advances in this area were made by Larsen and Lubotzky in [9].

The group G is said to have *polynomial representation growth* (PRG) if the sequence $R_N(G) := \sum_{n=1}^{N} r_n(G)$ is bounded by a polynomial. An important tool to study the representation growth of a PRG group G is its *representation zeta function*, viz. the Dirichlet series

$$\zeta_G(s) := \sum_{n=1}^{\infty} r_n(G) n^{-s},$$

where s is a complex variable. It is well-known that the *abscissa of convergence* $\alpha(G)$ of the series $\zeta_G(s)$, i.e. the infimum of all $\alpha \in \mathbb{R}$ such that $\zeta_G(s)$ converges on the complex right half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$, gives the precise degree of polynomial growth: $R_N(G) = O(1 + N^{\alpha(G)+\varepsilon})$ for every $\varepsilon \in \mathbb{R}_{>0}$.

In [2] we introduce new methods from the theory of $\mathfrak{p}$-adic integration to study representation zeta functions associated to compact p-adic analytic groups and arithmetic groups. In [3] we compute explicit formulae for the representation zeta functions of the groups $\mathrm{SL}_3(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of characteristic 0, in the case that the residue field characteristic is large compared to the ramification index of $\mathfrak{o}$.

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We give a summary of the main results of our forthcoming papers in the current section, followed by a brief description of the methodology in Section 2.

A finitely generated profinite group G is rigid if and only if it is FAb, i.e. if every open subgroup of G has finite abelianisation. In [8], Jaikin-Zapirain proved rationality results for the representation zeta functions of FAb compact p-adic analytic groups using tools from model theory. In particular, the representation zeta function of a FAb p-adic analytic pro-p group is a rational function in p^{-s}, for $p > 2$. Key examples of FAb compact p-adic analytic groups are the special linear groups $\mathrm{SL}_n(\mathfrak{o})$ and their principal congruence subgroups $\mathrm{SL}_n^m(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of characteristic 0 and residue field characteristic p. For fixed n, and varying m and $\mathfrak{o}$, the latter also yield important examples of families of pro-p groups which arise from a global Lie lattice, in this case $\mathfrak{sl}_n(\mathbb{Z})$.

To be more precise, let $\mathcal{O}$ be the ring of integers of a number field k, and let Λ be an $\mathcal{O}$-Lie lattice such that $k \otimes_{\mathcal{O}} \Lambda$ is a perfect k-Lie algebra of dimension d. Let $\mathfrak{o} = \mathcal{O}_v$ be the ring of integers of the completion k_v of k at a non-archimedean place v, lying above a rational prime p. Given a finite extension $\mathfrak{O}$ of $\mathfrak{o}$, we write $\mathfrak{P}$ for the maximal ideal of $\mathfrak{O}$, $e(\mathfrak{O}|\mathfrak{o})$ for the ramification index and $f(\mathfrak{O}|\mathfrak{o})$ for the residue class field extension degree. Let $\mathfrak{g}(\mathfrak{O}) := \mathfrak{O} \otimes_{\mathcal{O}} \Lambda$. For all sufficiently large m, the Lie lattice $\mathfrak{g}^m(\mathfrak{O}) := \mathfrak{P}^m\, \mathfrak{g}(\mathfrak{O})$ corresponds, by p-adic Lie theory, to a FAb, potent, saturable pro-p group $\mathsf{G}^m(\mathfrak{O}) := \exp(\mathfrak{g}^m(\mathfrak{O}))$. We call such m *permissible* for the Lie lattice $\mathfrak{g}(\mathfrak{O})$. For example, for unramified extensions $\mathfrak{O}$ of $\mathbb{Z}_p$ and p odd, every $m \in \mathbb{N}$ is permissible. In [2], we prove

Theorem A. *In the above setup, there exist a finite set S of places of k, a natural number r and a rational function $W(X_1, \ldots, X_r, Y) \in \mathbb{Q}(X_1, \ldots, X_r, Y)$ such that, for every non-archimedean place v of k with $v \notin S$, the following is true.*

There exist algebraic integers $\lambda_1 = \lambda_1(v), \ldots, \lambda_r = \lambda_r(v)$ such that for all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_v$ and for all $m \in \mathbb{N}$ which are permissible for $\mathfrak{g}(\mathfrak{O})$ one has

$$\zeta_{\mathsf{G}^m(\mathfrak{O})}(s) = q_v^{fdm}\, W(\lambda_1^f, \ldots, \lambda_r^f, q_v^{-fs}), \tag{1}$$

where q_v denotes the residue field cardinality of $\mathfrak{o}$, $f = f(\mathfrak{O}|\mathfrak{o})$ and $d = \mathrm{rank}_{\mathfrak{O}}(\mathfrak{g}(\mathfrak{O})) = \dim_k(k \otimes_{\mathcal{O}} \Lambda)$. Furthermore, the functional equation

$$\zeta_{\mathsf{G}^m(\mathfrak{O})}(s)\big|_{\substack{q_v \to q_v^{-1} \\ \lambda_i \to \lambda_i^{-1}}} = q_v^{fd(1-2m)} \zeta_{\mathsf{G}^m(\mathfrak{O})}(s) \tag{2}$$

holds.

Our proof of Theorem A implies in particular that the real parts of the poles of the zeta functions $\zeta_{\mathsf{G}^m(\mathfrak{O})}(s)$ are rational numbers. More precisely, we prove the following.

Theorem B. *In the above setup, there exists a finite set $P \subset \mathbb{Q}$ such that for all non-archimedean places v of k, all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_v$, and all permissible m for $\mathfrak{g}(\mathfrak{O})$ one has*

$$\{\mathrm{Re}(s) \mid s \textit{ a pole of } \zeta_{\mathsf{G}^m(\mathfrak{O})}(s)\} \subseteq P.$$

Furthermore, if $v \notin S$, where S is a finite set of places arising from Theorem A, and if $\mathfrak{o}_v \subseteq \mathfrak{O}_1 \subseteq \mathfrak{O}_2$, then for every $m \in \mathbb{N}$ which is permissible for $\mathfrak{g}(\mathfrak{O}_1)$ and $\mathfrak{g}(\mathfrak{O}_2)$,

$$\alpha(\mathsf{G}^m(\mathfrak{O}_1)) \leq \alpha(\mathsf{G}^m(\mathfrak{O}_2)). \tag{3}$$

Notice that, if the groups $\mathsf{G}^m(\mathfrak{O})$ are principal congruence subgroups of a FAb compact p-adic analytic group $\mathsf{G}(\mathfrak{O})$ consisting of the $\mathfrak{O}$-points of an algebraic group $\mathbf{G}$, such as $\mathbf{G} = \mathrm{SL}_n$, then (3) implies the monotonicity of the abscissae of convergence $\alpha(\mathsf{G}(\mathfrak{O}))$ under ring extensions. This follows from the fact that these abscissae are commensurability invariants. The set P of candidate poles is obtained by means of a resolution of singularities which leads to the generic formula (1). Theorems A and B are illustrated by the explicit formulae given in Theorem E below.

The arithmetic groups we are interested in are arithmetic lattices in semisimple algebraic groups defined over number fields. More precisely, let $\mathbf{G}$ be a connected, simply connected semisimple algebraic group, defined over a number field k, together with a fixed k-embedding into some GL_n. Let $\mathfrak{o}_S$ denote the ring of S-integers in k, for a finite set S of places of k including all the archimedean ones. We consider groups Γ which are commensurable to $\mathbf{G}(\mathfrak{o}_S) = \mathbf{G}(k) \cap \mathrm{GL}_n(\mathfrak{o}_S)$. In [10], Lubotzky and Martin showed that such a group Γ has PRG if and only if Γ has the Congruence Subgroup Property (CSP). Suppose that Γ has these properties. Then, according to a result of Larsen and Lubotzky [9, Proposition 1.3], the representation zeta function of Γ admits an Euler product decomposition. Indeed, if $\Gamma = \mathbf{G}(\mathfrak{o}_S)$ and if the congruence kernel of Γ is trivial, this decomposition is particularly easy to state: it takes the form

$$\zeta_\Gamma(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{[k:\mathbb{Q}]} \cdot \prod_{v \notin S} \zeta_{\mathbf{G}(\mathfrak{o}_v)}(s). \tag{4}$$

Here each archimedean factor $\zeta_{\mathbf{G}(\mathbb{C})}(s)$ enumerates rational representations of the group $\mathbf{G}(\mathbb{C})$; their contribution to the Euler product reflects Margulis super-rigidity. The groups $\mathbf{G}(\mathfrak{o}_v)$ are FAb compact p-adic analytic groups whose principal congruence subgroups fit into the framework of Theorems A and B; the product of the zeta functions of these local groups captures the finite image representations of Γ.

Several of the key results of [9] concern the abscissae of convergence of the 'local' representation zeta functions occurring as Euler factors on the right hand side of (4). With regards to abscissae of convergence of the 'global' representation zeta functions Avni proved that, for an arithmetic group Γ with the CSP, the abscissa of convergence of $\zeta_\Gamma(s)$ is always a rational number; see [1]. In [9, Conjecture 1.5], Larsen and Lubotzky conjectured that, for any two irreducible lattices Γ_1 and Γ_2 in a higher-rank semisimple group H, one has $\alpha(\Gamma_1) = \alpha(\Gamma_2)$, i.e. that the abscissa of convergence only depends on the ambient group. This can be regarded as a refinement of Serre's conjecture on the Congruence Subgroup Property. In [9, Theorem 10.1], Larsen and Lubotzky prove their conjecture in the case that H is a product of simple groups of type A_1, assuming Serre's conjecture. In [2], we prove

Theorem C. *Let Γ be an arithmetic lattice of a connected, simply connected simple algebraic group of type A_2 defined over a number field. If Γ has the CSP, then $\alpha(\Gamma) = 1$.*

Corollary D. *Assuming Serre's conjecture, Larsen and Lubotzky's conjecture holds for groups of the form* $H = \prod_{i=1}^{r} \mathbf{G}_i(K_i)$, *where each* K_i *is a local field of characteristic* 0 *and each* $\mathbf{G}_i$ *is an absolutely almost simple* K_i*-group of type* A_2 *such that* $\sum_{i=1}^{r} \mathrm{rk}_{K_i}(\mathbf{G}_i) \geq 2$ *and none of the* $\mathbf{G}_i(K_i)$ *is compact.*

Key to our proof of Theorem C is the following local result, which we formulate in accordance with the notation introduced before Theorem A.

Theorem E. *Let* $\mathfrak{o}$ *be a compact discrete valuation ring of characteristic* 0, *with residue field of cardinality* q. *Let* $\mathfrak{g}(\mathfrak{o})$ *be one of the following two* $\mathfrak{o}$*-Lie lattices of type* A_2*:*

(a) $\mathfrak{sl}_3(\mathfrak{o}) = \{\mathbf{x} \in \mathfrak{gl}_3(\mathfrak{o}) \mid \mathrm{Tr}(\mathbf{x}) = 0\}$*;*
(b) $\mathfrak{su}_3(\mathfrak{O}, \mathfrak{o}) = \{\mathbf{x} \in \mathfrak{sl}_3(\mathfrak{O}) \mid \mathbf{x}^\sigma = -\mathbf{x}^{\mathrm{t}}\}$, *where* $\mathfrak{O}|\mathfrak{o}$ *is an unramified quadratic extension with nontrivial automorphism* σ*.*

For $m \in \mathbb{N}$, *let* $\mathsf{G}^m(\mathfrak{o})$ *be the* m*-th principal congruence subgroup of the corresponding group* $\mathrm{SL}_3(\mathfrak{o})$ *or* $\mathrm{SU}_3(\mathfrak{O}, \mathfrak{o})$. *Assume that the residue field characteristic of* $\mathfrak{o}$ *is not equal to* 3. *Then, for all* $m \in \mathbb{N}$ *which are permissible for* $\mathfrak{g}(\mathfrak{o})$, *one has*

$$\zeta_{\mathsf{G}^m(\mathfrak{o})}(s) = q^{8m} \frac{1 + u(q)q^{-3-2s} + u(q^{-1})q^{-2-3s} + q^{-5-5s}}{(1 - q^{1-2s})(1 - q^{2-3s})},$$

where

$$u(X) = \begin{cases} X^3 + X^2 - X - 1 - X^{-1} & \text{if } \mathfrak{g}(\mathfrak{o}) = \mathfrak{sl}_3(\mathfrak{o}), \\ -X^3 + X^2 - X + 1 - X^{-1} & \text{if } \mathfrak{g}(\mathfrak{o}) = \mathfrak{su}_3(\mathfrak{O}, \mathfrak{o}). \end{cases}$$

The close resemblance between the representation zeta functions of the special linear and the special unitary groups is noteworthy and reminiscent of the Ennola duality for the characters of the corresponding finite groups of Lie type. We also give an explicit formula for $\zeta_{\mathrm{SL}_3^m(\mathfrak{o})}(s)$ in the exceptional case where $\mathfrak{o}$ is unramified and has residue field characteristic 3. Note that Theorem E implies that the abscissae of convergence $\alpha(\mathrm{SL}_3(\mathfrak{o}))$ and $\alpha(\mathrm{SU}_3(\mathfrak{O}, \mathfrak{o}))$ are each equal to $2/3$, as the abscissa of convergence is a commensurability invariant.

The explicit formula for $\zeta_{\mathrm{SL}_3(\mathfrak{o})}(s)$, which we present in [3], is deduced by means of the Kirillov orbit method, a description of the similarity classes in finite quotients of $\mathfrak{gl}_3(\mathfrak{o})$, and Clifford theory.

Theorem F. *There exist finitely many polynomials* $f_{\tau,i}, g_{\tau,i} \in \mathbb{Q}[x]$, *indexed by* $(\tau, i) \in \{1, -1\} \times I$, *such that for every compact discrete valuation ring* $\mathfrak{o}$ *of characteristic* 0, *with residue field characteristic* $p > 3e(\mathfrak{o}|\mathbb{Z}_p)$, *one has*

$$\zeta_{\mathrm{SL}_3(\mathfrak{o})}(s) = \frac{\sum_{i \in I} f_{\tau,i}(q)(g_{\tau,i}(q))^{-s}}{(1 - q^{1-2s})(1 - q^{2-3s})},$$

where q *denotes the size of the residue field of* $\mathfrak{o}$ *and* $q \equiv \tau \pmod 3$.

An explicit set of such polynomials $f_{\tau,i}, g_{\tau,i}$ is computed in [3]. This result should be seen against the background of [8, Theorem 1.1], which establishes the rationality of representation zeta functions of FAb compact p-adic analytic groups. For groups of the form $\mathrm{SL}_3(\mathfrak{o})$, Theorem F specifies that these rational functions vary 'uniformly' as a function of the residue field cardinality q. Theorem F enables us to analyse the global representation zeta functions of the arithmetic groups $\mathrm{SL}_3(\mathcal{O}_S)$.

Theorem G. *Let $\mathfrak{o}_S$ be the ring of S-integers of a number field k, where S is a finite set of places of k including all the archimedean ones. Then there exists $\varepsilon > 0$ such that the representation zeta function of $\mathrm{SL}_3(\mathfrak{o}_S)$ admits a meromorphic continuation to the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 - \varepsilon\}$. The continued function is holomorphic on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1\}$ except for a double pole at $s = 1$. There is a constant $c \in \mathbb{R}_{>0}$ such that*

$$\sum_{n=1}^{N} r_n(\mathrm{SL}_3(\mathfrak{o}_S)) \sim c \cdot N \log N,$$

where $f(N) \sim g(N)$ means $\lim_{N\to\infty} f(N)/g(N) = 1$.

In [2], we also give simple geometric estimates for the abscissae of convergence of representation zeta functions of compact p-adic analytic groups and we compute representation zeta functions associated to norm-1 groups in non-split quaternion algebras.

2. Methodology

2.1. **Kirillov orbit method and $\mathfrak{p}$-adic integration.** The core technique in [2] is a $\mathfrak{p}$-adic formalism for the representation zeta functions of potent, saturable pro-p groups. This approach has two key ingredients. Firstly, the *Kirillov orbit method* for potent, saturable pro-p groups provides a way to construct the characters of these groups in terms of co-adjoint orbits; cf., e.g., [5]. This 'linearisation' – pioneered in [6, 8] – allows us to transform the original problem of enumerating representations by their dimension into the problem of counting co-adjoint orbits by their size. The second main idea of our approach is to tackle the latter problem with the help of suitable $\mathfrak{p}$-*adic integrals* which are closely related to Igusa local zeta functions (cf. [4, 7, 11]) and conceptually simpler than the definable integrals utilised in [8].

For pro-p groups of the form $\mathsf{G}^m(\mathfrak{O})$, which arise from a global $\mathfrak{o}$-Lie lattice Λ as in the setup of Theorems A and B, we describe the representation zeta functions in terms of $\mathfrak{p}$-adic integrals of the shape

$$\mathcal{Z}_{\mathfrak{O}}(r,t) = \int_{(x,\mathbf{y})\in V(\mathfrak{O})} |x|_{\mathfrak{P}}^t \prod_{1\le j\le \lfloor d/2\rfloor} \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y})x^2\|_{\mathfrak{P}}^r}{\|F_{j-1}(\mathbf{y})\|_{\mathfrak{P}}^r}\, d\mu(x,\mathbf{y}). \tag{5}$$

Here $\mathfrak{P}$ denotes the maximal ideal of the discrete valuation ring $\mathfrak{O}$, the domain of integration $V(\mathfrak{O}) \subset \mathfrak{O}^{d+1}$ is a union of cosets modulo $\mathfrak{P}$, the additive Haar measure μ is normalised so that $\mu(\mathfrak{O}^{d+1}) = 1$, the $F_j(\mathbf{y})$ are families of polynomials over the global ring $\mathfrak{o}$, which may be defined in terms of the structure constants of the lattice Λ with respect to a given $\mathfrak{o}$-basis, and we write $\|\cdot\|_{\mathfrak{P}}$ for the $\mathfrak{P}$-adic maximum norm.

The link between the Kirillov orbit formalism and the $\mathfrak{p}$-adic integrals (5) is given by the fact that the problem of enumerating finite co-adjoint orbits of given size may be reformulated as the problem of enumerating elementary divisors of matrices of linear forms. The integrals (5) are multivariate analogues of Igusa local zeta functions associated to polynomial mappings, a well-studied class of local zeta integrals; see [11]. They are also akin to the $\mathfrak{p}$-adic integrals studied in [12], and our proofs of Theorems A and B rely on an adaptation of the methods and formulae provided there.

A key point in the proof of Theorem C is the fact that, for the relevant arithmetic groups Γ, almost all of the non-archimedean Euler factors in (4) are of the form $\zeta_{\mathrm{SL}_3(\mathfrak{o})}(s)$ or $\zeta_{\mathrm{SU}_3(\mathfrak{O},\mathfrak{o})}(s)$, and that for the exceptional factors the abscissa of convergence is sufficiently small, in fact equal to 2/3. In [2] we use the exact formulae provided by Theorem E, together with Clifford theory and suitable approximative Dirichlet series, to prove that the global abscissa of convergence $\alpha(\Gamma)$ is always equal to 1.

The proof of Theorem E relies on a concrete interpretation of the $\mathfrak{p}$-adic integrals (5). More generally, in the case of 'semisimple' compact p-adic analytic groups, we give a description of these integrals in terms of a filtration of the irregular locus of the associated Lie algebra. The terms of this filtration are projective algebraic varieties defined by centraliser dimension, refining the notion of irregularity. In the case of $\mathfrak{sl}_3$ and $\mathfrak{su}_3$, this filtration is simple enough to allow explicit computations.

2.2. **Similarity classes, shadows and Clifford theory.** The explicit computation of the zeta function of $\mathrm{SL}_3(\mathfrak{o})$ in Theorem F is based on the Kirillov orbit method and a quantitative analysis of the $\mathrm{GL}_3(\mathfrak{o})$-adjoint orbits, or *similarity classes*, in the finite quotients $\mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell)$, $\ell \in \mathbb{N}$. The pre-images of similarity classes in $\mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell)$ under the natural reduction map $\pi_\ell : \mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^{\ell+1}) \to \mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell)$ are unions of similarity classes. In [3] we give an explicit, recursive description of these classes which allow us to compute their numbers and cardinalities.

Given $\ell \in \mathbb{N}$ and a similarity class $\mathcal{C} \in \mathrm{GL}_3(\mathfrak{o})\backslash\mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell)$, we define the shadow $\mathrm{sh}(\mathcal{C})$ of $\mathcal{C}$ to be the conjugacy class of the reduction modulo $\mathfrak{p}$ of $\mathrm{Stab}_{\mathrm{GL}_3(\mathfrak{o})}(A)$, where A is any element of $\mathcal{C}$. This definition is independent of the choice of A. We write $\mathfrak{Sh} := \{\mathrm{sh}(\mathcal{C}) \mid \mathcal{C} \in \mathrm{GL}_3(\mathfrak{o})\backslash\mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell), \ell \in \mathbb{N}\}$ for the set of shadows. It turns out that $|\mathfrak{Sh}| = 10$, and that the shadow of a similarity class $\mathcal{C}$ determines the numbers and sizes of the classes which make up $\pi_\ell^{-1}(\mathcal{C})$. More precisely, there are polynomials $a_{\sigma_1,\sigma_2}(x), b_{\sigma_1,\sigma_2}(x) \in \mathbb{Q}[x]$, indexed by $(\sigma_1, \sigma_2) \in \mathfrak{Sh}^2$, such that, for all $\ell \in \mathbb{N}$, any similarity class $\mathcal{C} \in \mathrm{GL}_3(\mathfrak{o})\backslash\mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell)$ and shadow $\sigma \in \mathfrak{Sh}$, the following hold:

(a) The number of similarity classes in $\pi_\ell^{-1}(\mathcal{C})$ with shadow σ is equal to $a_{\mathrm{sh}(\mathcal{C}),\sigma}(q)$.
(b) All similarity classes in $\pi_\ell^{-1}(\mathcal{C})$ with shadow σ have equal cardinality $|\mathcal{C}|\cdot b_{\mathrm{sh}(\mathcal{C}),\sigma}(q)$.

We then compute, for each $\ell \in \mathbb{N}$ and $\sigma \in \mathfrak{Sh}$, the Dirichlet generating function which enumerates the classes with shadow σ, viz.

$$\zeta_\ell^\sigma(s) := \sum_{\mathcal{C}\in\mathcal{Q}_\ell[\sigma]} |\mathcal{C}|^{-s}, \quad \text{where } \mathcal{Q}_\ell[\sigma] := \{\mathcal{C} \in \mathrm{GL}_3(\mathfrak{o})\backslash\mathfrak{gl}_3(\mathfrak{o}/\mathfrak{p}^\ell) \mid \mathrm{sh}(\mathcal{C}) = \sigma\}.$$

Finally, we prove that if $p > 3e(\mathfrak{o}|\mathbb{Z}_p)$ then

$$\zeta_{\mathrm{SL}_3(\mathfrak{o})}(s) = \lim_{\ell\to\infty} |\mathfrak{o}/\mathfrak{p}|^{-\ell} \sum_{\sigma\in\mathfrak{Sh}} \frac{|\mathrm{GL}_3(\mathfrak{o}/\mathfrak{p}) : H(\sigma)|^{1-s/2}\, \zeta_{H(\sigma)\cap\mathrm{SL}_3(\mathfrak{o}/\mathfrak{p})}(s)}{|\mathrm{SL}_3(\mathfrak{o}/\mathfrak{p}) : H(\sigma)\cap\mathrm{SL}_3(\mathfrak{o}/\mathfrak{p})|}\, \zeta_\ell^\sigma(s/2), \tag{6}$$

where, for each $\sigma \in \mathfrak{Sh}$, $H(\sigma) \leq \mathrm{GL}_3(\mathfrak{o}/\mathfrak{p})$ denotes a fixed representative of the conjugacy class σ. Theorem F is a consequence of (6). Theorem G follows from an analysis of the formula given in Theorem F, and standard Tauberian theorems.

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